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SURFACE CURVES FOR STEADY NONUNIFORM FLOW

By Robert B. Jansen, Jun. ASCE

HYDRAULICS DIVISION

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

SURFACE CURVES FOR STEADY NONUNIFORM FLOW

By Robert B. Jansen,[1] Jun. ASCE

Synopsis

Few problems in the field of hydraulic research have been more challenging or baffling than that concerned with plotting the surface curve assumed by a liquid in varied or nonuniform flow. Several investigators have sought the solution by attempting to integrate the varied flow equation. This paper, however, makes use of the differential equation by direct application of the surface slope that it defines. The paper is confined to the study of steady flow in prismatic trapezoidal channels, with the rectangular section considered to be a limiting case. Surface profiles plotted by this method are compared with those obtained by integration.

Introduction

A quick and simple method of determining the water surface profile in open channels carrying nonuniform flow has been the object of a search by engineers and mathematicians for a number of years. Although many exhaustive efforts have been made to solve this problem by integration, no entirely satisfactory answer has been found. The method advocated by Boris A. Bakhmeteff,[2] Hon. M. ASCE, has been applied with more confidence than has been accorded the works of other researchers, but even this approach has its deficiencies that are generally recognized.

The procedure outlined herein is, regrettably, not intended to be a final and complete solution. It is certainly not presented as a substitute for the scholarly achievements of Mr. Bakhmeteff, nor to detract from the works of other earnest investigators. The paper is essentially an enlargement upon the obvious relationship defined by the differential equation of varied flow, and is

Note.—Written comments are invited for publication; the last discussion should be submitted by April 1, 1952.

[1] Civ. Eng. Associate, Bureau of Engineering, City of Los Angeles, Calif.

[2] "Hydraulics of Open Channels," B. A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y. 1932.

offered as illustration of the results obtainable by direct plotting of the water surface slope.

Equation of Varied Flow

The flow of a liquid in an open channel is said to be uniform when factors such as bottom slope, depth, velocity, and cross-sectional area remain constant from point to point. Uniform motion can be considered to be a limit that is approached, for example, in a long reach of channel whose slope is just equal to the rate of loss in energy. In general, therefore, the case that must be considered is the one in which the contributing factors vary in such a manner that the energy gradient, the bottom slope, and consequently the water surface are not parallel.

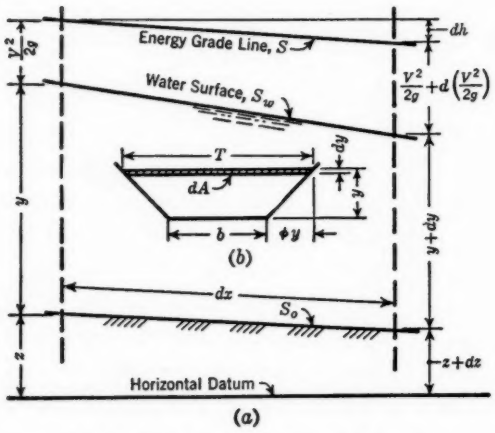

Fig. 1.—Flow in a Trapezoidal Channel

The following treatment is devoted to steady nonuniform (or varied) flow in prismatic channels. The slope of streamlines is assumed to be either constant or varying so gradually that the effect of centrifugal action can be neglected. That is, the pressure distribution is taken as hydrostatic. In the calculation of energy losses, application is made of the Manning version of the Chezy formula, that entails the assumption that the energy gradient in varied flow is parallel to that corresponding to uniform flow under the same conditions of depth, roughness, and velocity.

If x is the distance along the channel parallel to the bottom for an elementary length of channel (dx), as illustrated in Fig. 1(a), the total head (h) at any point in the stream is

$$h = z + y + \frac{V^2}{2g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

assuming the velocity (V) to be constant with respect to depth. In Eq. 1, z is the elevation of the channel bottom above a horizontal datum, y is the depth of water in the channel, and g is the acceleration of gravity.

The derivative of h with respect to x is

$$\frac{dh}{dx} = \frac{dz}{dx} + \frac{dy}{dx} + \frac{d}{dx}\left(\frac{V^2}{2g}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and $\frac{dh}{dx} = -S = -\frac{V^2}{C^2 R}$ from the Chezy formula, wherein $V = C\sqrt{RS}$, in which C is a coefficient; S is the slope of the energy gradient; and R is the

hydraulic radius. Also, from Fig. 1(*a*) it is apparent that $\frac{dz}{dx} = -S_o$, in which S_o is the slope of the channel bottom. Referring to Fig. 1(*b*) if infinitesimals higher than the first order are neglected, $dA = T\,dy$, in which T is the surface width and A is the cross-sectional area of the channel.

Then, $\frac{d}{dx}\left(\frac{V^2}{2g}\right) = \frac{V}{g}\left(\frac{dV}{dx}\right) = \frac{Q}{Ag}\left(\frac{dV}{dx}\right)$, and since $\frac{dV}{dx} = \frac{d}{dx}\left(\frac{Q}{A}\right)$ $= -\frac{Q}{A^2}\left(\frac{dA}{dx}\right)$, it follows that

$$\frac{d}{dx}\left(\frac{V^2}{2g}\right) = \frac{-Q^2}{A^3 g}\frac{dA}{dx} = \frac{-Q^2 T}{A^3 g}\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Substituting those derived terms into Eq. 2 gives

$$\frac{-V^2}{C^2 R} = -S_o + \frac{dy}{dx} - \frac{Q^2 T}{A^3 g}\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

By transposing terms, the following final expression for the slope of the water surface is obtained:

$$\frac{dy}{dx} = \frac{S_o - \dfrac{Q^2}{A^2 C^2 R}}{1 - \dfrac{Q^2 T}{A^3 g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now, if, as indicated in Fig. 1(*b*), the tangent of the channel side slope is designated as ϕ, then the area (A) is $A = y\,(b + \phi\, y) = b\,y + \phi\, y^2$, in which b is the bottom width of the channel, and the surface width is $T = b + 2\phi\, y$. Also, the wetted perimeter is $p = b + 2\sqrt{\phi^2 y^2} + y^2 = b + 2\,y\sqrt{1 + \phi^2}$. The expression for the hydraulic radius is therefore

$$R = \frac{b\,y + \phi\, y^2}{b + 2\,y\sqrt{1 + \phi^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

For a prismatic channel, S_o is constant. By the Manning formula, $C = \frac{1.486}{n} R^{\frac{1}{6}}$, squaring, $C^2 = \frac{2.208\, R^{\frac{1}{3}}}{n^2}$.

By inserting the foregoing factors, Eq. 5 can now be written

$$\frac{dy}{dx} = \frac{S_o - \dfrac{Q^2}{(b\,y + \phi\, y^2)^2\left(\dfrac{2.208\, R^{\frac{1}{3}}}{n^2}\right)(R)}}{1 - \dfrac{Q^2}{g}\left[\dfrac{b + 2\phi\, y}{(b\,y + \phi\, y^2)^3}\right]} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

or,

$$\frac{dy}{dx} = \frac{S_o - \left(\frac{n^2 Q^2}{2.208}\right)\left[\frac{(b + 2y)\sqrt{1 + \phi^2})^{\frac{4}{3}}}{(b y + \phi y^2)^{\frac{10}{3}}}\right]}{1 - \frac{Q^2}{g}\left[\frac{b + 2\phi y}{(b y + \phi y^2)^3}\right]} \quad \ldots\ldots\ldots (7b)$$

The constant terms in Eq. 7*b* can be denoted as

$$N = \frac{n^2 Q^2}{2.208}; \qquad P = 2\sqrt{1 + \phi^2}; \qquad \text{and} \qquad G = \frac{Q^2}{g}.$$

By the indicated substitutions, the differential equation can be changed to the form

$$\frac{dy}{dx} = \frac{S_o - N\left[\frac{(b + P y)^{\frac{4}{3}}}{(b y + \phi y^2)^{\frac{10}{3}}}\right]}{1 - G\left[\frac{b + 2\phi y}{(b y + \phi y^2)^3}\right]} \quad \ldots\ldots\ldots (8)$$

For the special case of a rectangular cross section, $\phi = 0$, and $P = 2$ so that, for this shape,

$$\frac{dy}{dx} = \frac{S_o - N\left[\frac{(b + 2y)^{\frac{4}{3}}}{(b y)^{\frac{10}{3}}}\right]}{1 - G\left[\frac{b}{(b y)^3}\right]} \quad \ldots\ldots\ldots (9)$$

Since for a prismatic channel the terms b, ϕ, and P do not vary, the expressions within brackets of Eq. 8 are functions of the depth alone. These variables in the numerator and denominator are designated as u and j, respectively.

$$u = \frac{(b + P y)^{\frac{4}{3}}}{(b y + \phi y^2)^{\frac{10}{3}}} \quad \ldots\ldots\ldots (10)$$

and

$$j = \frac{b + 2\phi y}{(b y + \phi y^2)^3} \quad \ldots\ldots\ldots (11)$$

The relation for the trapezoidal section can now be restated as

$$\frac{dy}{dx} = \frac{S_o - N u}{1 - G j} \quad \ldots\ldots\ldots (12)$$

The depth functions, u and j, have been evaluated and tabulated for common cases of side walls.[2a]

Computation Procedure

To illustrate the direct use of the surface slope in solving problems of varied flow, the following elementary examples are given:

[2a] Tables for the values of the factors u and j are available for the cases of vertical side walls and side walls having a slope of 1 on 1 and 1 on 1.5 with bottom widths ranging from 5 ft to 50 ft in 5-ft intervals and depths from 1 ft to 10 ft in 0.5-ft intervals. The cost of these tables is $1.00.

Example 1.—Water is flowing from under a sluice into a rectangular canal whose bottom width is 20 ft. The channel grade is 0.0010 and the value of Manning's n is 0.013. The sluice is regulated to discharge 1,080 cu ft per sec with a depth at the vena contracta of 1.5 ft. The surface curve is required.

TABLE 1.—DETERMINATION OF SURFACE SLOPE, EXAMPLE 1

Depth y	Factor u	Factor j	Slope $\frac{dy}{dx}$
2.0	316.3×10^{-6}	312.5×10^{-6}	+0.00264
2.5	158.7	160.0	+0.00275
3.0	91.09	92.59	+0.00303
3.5	57.30	58.31	+0.00371
4.0	38.54	39.06	+0.00591

The critical depth is found to be 4.49 ft. Flow under the conditions stated will usually lead to an hydraulic jump, starting between $y = 1.5$ and a critical depth $y_c = 4.49$. For the purpose of illustration, however, the curve will be computed on the assumption that the tailwater is low enough to prevent occurrence of the jump.

Evaluating the terms in Eq. 12, $S_o = 0.0010$, $N = \frac{n^2 Q^2}{2.208} = \frac{(0.013)^2(1080)^2}{2.208}$ $= 89.3$, $G = \frac{Q^2}{g} = \frac{(1080)^2}{32.2} = 36{,}200$. From tables or by substitution in Eqs. 10 and 11, the values of u and j can be found. Solution of Eq. 12 is then completed.

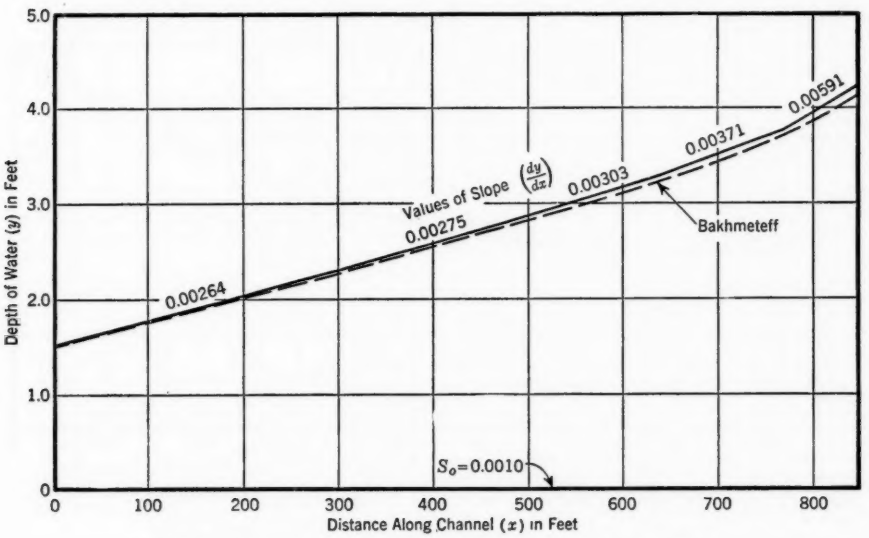

FIG. 2.—DETERMINATION OF WATER SURFACE SLOPE—EXAMPLE 1

It should be remembered that $\frac{dy}{dx}$ is the water slope with respect to the channel bottom. By conventions of the calculus, its sign is positive with increasing depth in the direction of flow.

In the manner described above, values of $\frac{dy}{dx}$ for intermediate depths have been determined and are listed in Table 1.

The water surface curve described by these slopes is plotted in Fig. 2. In order to attain uniform results in the sketching of curves in this paper the tangents have been made to intersect at the mid-points of the depth increments, that is, at $y = 2.25$, $y = 2.75$, and so on. In practice, this procedure can be modified to give more accuracy in that part of the curve where $\frac{dy}{dx}$ is varying rapidly.

TABLE 2.—DETERMINATION OF SURFACE SLOPE, EXAMPLE 2

Depth y	Factor u	Factor j	Slope $\frac{dy}{dx}$
2.0	8689×10^{-6}	5000×10^{-6}	+0.0009
2.5	4753	2560	+0.0020
3.0	2939	1481	+0.0024
3.5	1974	932.9	+0.0026
4.0	1408	625.0	+0.0027

For comparison, the surface plot obtained by the Bakhmeteff method is shown by the dashed curve in Fig. 2.

Example 2.—In an experiment conducted by the University of Illinois Engineering Experiment Station,[3] Urbana, Ill., 40.39 cu ft per sec of water was caused to flow through a rectangular channel 5 ft wide. The flume was laid on a slope of $S_o = 0.0030$ and was artificially roughened to obtain a value of $n = 0.019$. A backwater gate installed at the lower end of the reach was

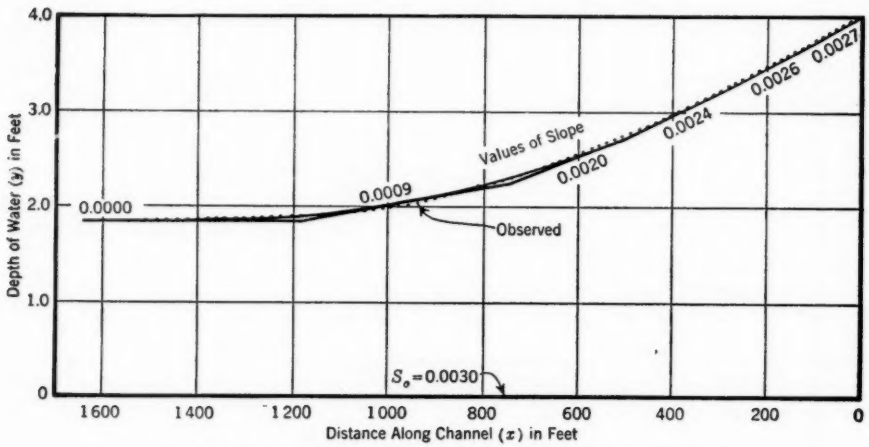

FIG. 3.—DETERMINATION OF WATER SURFACE SLOPE—EXAMPLE 2

regulated in this particular test so that the depth at the downstream gage was $y = 4.0$ ft. The normal depth (y_o) for the given conditions is found to equal 1.84 ft. Also, using procedures already described, $N = 0.267$ and $G = 50.7$.

[3] "An Investigation of the Backwater Profile for Steady Flow in Prismatic Channels," by W. M. Lansford and W. D. Mitchell, *Bulletin Series No. 381*, Univ. of Illinois, Urbana, Ill., 1949.

The slope calculations are summarized in Table 2.

In Fig. 3 the surface plotted by means of these tangents is shown as compared with the curve observed in the laboratory.

Conclusions

Direct application of the water surface slope as defined by the varied flow equation may be expected to yield results in close agreement with those obtained by accepted integration procedures and by trial-and-error methods. Any of these approaches involves an uncertain amount of inaccuracy due to the assumptions of hydrostatic pressure and of nondivergent motion. The error introduced through determination of energy loss by the uniform-flow formula is probably not appreciable except in relatively steep or sharply nonparallel movement.

Profiles plotted by any procedure are inaccurate in the vicinity of the control section for most cases, since in the vicinity of the critical depth the effect of centrifugal forces is usually of considerable importance. The water slope at this point, when channel conditions permit, can be approximated by a curve sketched between adjacent sections upstream and downstream.

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